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December 15, 1960

AEC Research and Development Report

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ABSTRACT

As an extension of an earlier report the Green's function is evaluated for the Lundquist equations linearized about uniform magnetic field, constant matter density and zero flow velocity. It is assumed that all quantities are functions of two space variables and time only. In the general magnetic field configuration considered here a pure Alfven disturbance no longer exists; there is instead a wave with properties of both the Alfven and fast-slow disturbance.

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On the Green's Function for Two-Dimensional Magnetohydrodynamic Waves. II

In a previous paper[1] we studied the fundamental solution of the linearized Lundquist equations for a restricted two dimensional situation. One of the essential simplifications resulting from the assumptions made was the decoupling of the Alfven and fast-slow (or "longitudinal") waves. Here we wish to consider the most general two dimensional situation in which the Alfven waves are coupled to the fast-slow waves. For the "physical" part of the fundamental solution, we shall again give an algebraic function plus standard distributions.[2] We shall draw heavily on the material presented in I and omit all arguments which are parallels or direct extensions of those found there.

We carry over all the hypotheses of I but one, we now assume that the constant magnetic field need not lie in the $x\ y$ plane. Instead (and with somewhat irregular notation) we assume (compare I(2))

$$(1) \qquad \vec{B}_o = B_o(1,0,\epsilon) = B_o\vec{b}.$$

With this extension we now have the general two-dimensional situation. There is one other special two dimensional situation, where $\vec{B} = (0,0,B(x,y))$, which has been studied even for the non-linear

equations[3] and which we ignore. Besides the symbols in I we need one new speed similar to the Alfven speed:

$$(2) \qquad A_1^2 = \frac{B_o^2 (1 + \epsilon^2)}{\mu \rho_o} .$$

The equations of motion are just the same as I(1) and the equations for the transforms are the same as I(3); they are:

$$(3a) \qquad \omega\rho - (\vec{k}\cdot\vec{u}) = iS_\rho$$

$$(3b) \qquad \omega\vec{u} - a_o^2 \vec{k}\rho + A_o^2 \vec{b}\times(\vec{B}\times\vec{k}) = i\vec{S}_u$$

$$(3c) \qquad \omega\vec{B} + \vec{k}\times(\vec{u}\times\vec{b}) = i\vec{S}_B .$$

With $\vec{b}$ given its new meaning by (1), we may no longer split off the z components of $\vec{u}$ and $\vec{B}$ as in I. Instead we use a decomposition procedure essentially equivalent to that described by H. Grad.[4]

From (3) we derive

$$(4a) \qquad \omega(\vec{k}\cdot\vec{u}) - a_o^2 k^2\rho + A_o^2((\vec{B}\cdot\vec{k})(\vec{b}\cdot\vec{k}) - k^2(\vec{b}\cdot\vec{B})) = i\vec{k}\cdot\vec{S}_u$$

$$(4b) \qquad \omega(\vec{b}\cdot\vec{u}) - a_o^2(\vec{k}\cdot\vec{b})\rho = i\vec{b}\cdot\vec{S}_u$$

$$(4c) \qquad \omega(\vec{B}\cdot\vec{b}) + (\vec{u}\cdot\vec{b})(\vec{k}\cdot\vec{b}) - b^2(\vec{k}\cdot\vec{u}) = i\vec{b}\cdot\vec{S}_B$$

$$(4d) \qquad \omega(\vec{k}\cdot\vec{B}) = i\vec{k}\cdot\vec{S}_B,$$

and with (3a) we have a closed system for ρ, $(\vec{b}\cdot\vec{u})$, and $(\vec{b}\cdot\vec{B})$,

so that

$$(5a)\qquad \rho = \frac{i}{\tilde{D}}\left(\omega(\omega^2 - A_1^2k^2)S_\rho + \omega^2(\vec{k}\cdot\vec{S}_u) - k^2A_o^2(\vec{b}\cdot\vec{k})(\vec{b}\cdot\vec{S}_u)\right.$$

$$\left. + \omega A_o^2(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{S}_B)\right)$$

$$(5b)\qquad (\vec{u}\cdot\vec{b}) = \frac{i}{\tilde{D}}\left(\omega(\omega^2 - A_1^2k^2)(\vec{b}\cdot\vec{S}_u) + a_o^2\omega(\vec{b}\times\vec{k}\cdot\vec{k}\times\vec{S}_u)\right.$$

$$\left. + \vec{k}\cdot\vec{b}(\omega^2 - k^2A_1^2)S_\rho + a_o^2A_o^2\ \vec{b}\cdot\vec{k}(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{S}_B)\right)$$

$$(5c)\qquad (\vec{B}\cdot\vec{b}) = \frac{i}{\tilde{D}}\left(\omega(\omega^2 - k^2a_o^2)\vec{S}_B\cdot\vec{b} + \omega S_\rho a_o^2(k^2b^2 - .(\vec{k}\cdot\vec{b})^2)\right.$$

$$\left. + \omega^2(\vec{b}\times\vec{k}\cdot\vec{b}\times\vec{S}_u) + a_o^2\vec{b}\cdot\vec{k}(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{S}_u)\right)$$

$$+ \frac{i}{\omega\tilde{D}}(\vec{k}\cdot\vec{S}_B)(\vec{b}\cdot\vec{k})A_o^2\ (a_o^2(\vec{b}\cdot k)^2 - b^2\omega^2)$$

where

$$(6)\qquad \tilde{D} = \tilde{D}(\omega,\vec{k}) = \omega^4 - \omega^2(a_o^2 + A_1^2)k^2 + k^2(k_x)^2a_o^2A_o^2$$

(Compare with D as defined in I.) In (5c) the second term is the "unphysical" term. We may evaluate all the transforms in (5) directly by the methods of I.

If we can give $\vec{u}\times\vec{b}$ and $\vec{B}\times\vec{b}$, possibly in terms of ρ, $(\vec{u}\cdot\vec{b})$, $(\vec{B}\cdot\vec{b})$, then we do have ρ, $\vec{u}$ and $\vec{B}$, but again from (3) we derive

$$\text{(7a)} \quad \omega(\vec{u}\times\vec{b}) + A_o^2(\vec{B}\times\vec{b})(\vec{b}\cdot\vec{k}) = i(\vec{S}_u\times\vec{b}) + \vec{k}\times\vec{b}(a_o^2\rho + A_o^2(\vec{b}\cdot\vec{B}))$$

$$\text{(7b)} \quad \omega(\vec{B}\times\vec{b}) + (\vec{u}\times\vec{b})(\vec{b}\cdot\vec{k}) = i(\vec{S}_B\times\vec{b}).$$

It is possible to give many equivalent representations of the solutions of (7). We choose an unsymmetrical one which emphasizes certain properties of the solution. In the form chosen we see easily that for $\epsilon = 0$ the results coincide with I and thus we have separated the part of the Alfven wave that does not vanish as $\epsilon \to 0$. We define the vectors

$$\text{(8a)} \qquad \vec{j} = (0,1,0)$$

$$\text{(8b)} \qquad \vec{n} = (-\epsilon,0,1),$$

so that $\vec{b},\vec{j},\vec{n}$ form a right handed orthogonal set. With $(\vec{u}\cdot\vec{b})$ and $(\vec{B}\cdot\vec{b})$ given we need only specify the quantities $(\vec{u}\cdot\vec{j})$, $(\vec{u}\cdot\vec{n})$, $(\vec{B}\cdot\vec{j})$, $(\vec{B}\cdot\vec{n})$. With the definition

$$\text{(9)} \qquad \Delta = \omega^2 - A_o^2\,(\vec{k}\cdot\vec{b})^2$$

we find

$$\text{(10a)} \quad (\vec{u}\cdot\vec{j}) = \frac{i}{\tilde{D}}\Big\{a_o^2\omega((\vec{k}\cdot\vec{S}_u) + \omega S_\rho)(\vec{k}\cdot\vec{j}) + \omega(\omega^2 - a_o^2k^2)(\vec{S}_u\cdot\vec{j})$$
$$+ A_o^2\, a_o^2(\vec{k}\cdot\vec{b})(\vec{k}\cdot\vec{n})(\vec{b}\cdot\vec{k}\times\vec{S}_B) + (\vec{n}\cdot\vec{k}\times\vec{S}_B)A_o^2(\omega^2 - a_o^2(\vec{k}\cdot\vec{b})^2)\Big\}$$
$$+ \frac{i}{\Delta\tilde{D}}\,(\vec{n}\cdot\vec{k})A_o^2\omega^2\Big\{\omega(\vec{b}\cdot\vec{k}\times\vec{S}_u) - A_o^2(\vec{k}\cdot\vec{b})(\vec{k}\times\vec{S}_B\cdot\vec{b})\Big\}$$

$$(10b)\quad (\vec{u}\cdot\vec{n}) = \frac{i}{\Delta}\left\{(\vec{j}\cdot\vec{k}\times\vec{S}_B)A_o^2 + \vec{S}_u\cdot\vec{n}\omega\right\} + \frac{+i}{\tilde{D}}\left\{(b^2-1)(\vec{j}\cdot\vec{k}\times\vec{S}_B)\omega^2\right.$$

$$\left. - a_o^2(\vec{j}\cdot\vec{k})(\vec{n}\cdot\vec{k})(\vec{n}\cdot\vec{k}\times\vec{S}_B) + a_o^2\omega(\vec{k}\cdot\vec{n})(\vec{k}\cdot\vec{S}_u + \omega S\rho)\right\}$$

$$+ \frac{i\omega^2 A_o^2}{\Delta\tilde{D}}\left\{(b^2-1)(\vec{k}\cdot\vec{b})^2(\vec{j}\cdot\vec{k}\times\vec{S}_B) - b^2(\vec{n}\cdot\vec{k})(\vec{j}\cdot\vec{k})(\vec{n}\cdot\vec{k}\times\vec{S}_B)\right.$$

$$\left. + \omega(\vec{k}\cdot\vec{n})(\vec{b}\times\vec{k}\cdot\vec{b}\times\vec{S}_u)\right\}$$

$$(10c)\quad (\vec{B}\cdot\vec{j}) = \frac{i}{\tilde{D}}\left\{-a_o^2(\vec{b}\cdot\vec{k})(\vec{k}\cdot\vec{j})(\vec{k}\cdot\vec{S}_u + \omega S_\rho) + \omega(\vec{S}_B\cdot\vec{j})(\omega^2 - a_o^2k^2)\right.$$

$$\left. -(\vec{S}_u\cdot\vec{j})(\vec{b}\cdot\vec{k})(\omega^2 - a_o^2k^2)\right\}$$

$$- \frac{i\omega^2 A_o^2(\vec{k}\cdot\vec{n})}{\Delta\tilde{D}}\left\{\omega(\vec{S}_B\times\vec{k}\cdot\vec{b}) + (\vec{b}\cdot\vec{k})(\vec{b}\cdot\vec{k}\times\vec{S}_u)\right\}$$

$$+ \frac{i}{\omega\tilde{D}}(\vec{k}\cdot\vec{S}_B)\left\{A_o^2(\vec{k}\cdot\vec{j})(a_o^2(\vec{k}\cdot\vec{b})^2 - \omega^2b^2)\right\}$$

$$(10d)\quad (\vec{B}\cdot\vec{n}) = \frac{i}{\Delta}\left\{\omega(\vec{S}_B\cdot\vec{n}) - (\vec{b}\cdot\vec{k})(\vec{S}_u\cdot\vec{n})\right\} - \frac{i}{\tilde{D}}(\vec{k}\cdot\vec{n})(\vec{k}\cdot\vec{b})a_o^2(\vec{k}\cdot\vec{S}_u + \omega S_\rho)$$

$$+ \frac{i\omega^2 A_o^2}{\Delta\tilde{D}}\left\{-(\vec{k}\cdot\vec{n})(\vec{k}\cdot\vec{b})(\vec{b}\times\vec{k}\cdot\vec{b}\times\vec{S}_u) + \omega(b^2-1)k^2(\vec{S}_B\cdot\vec{n})\right.$$

$$\left. + \omega(\vec{S}_B\cdot\vec{j})(\vec{k}\cdot\vec{n})(\vec{k}\cdot\vec{j})\right\} + \frac{i(\vec{k}\cdot\vec{S}_B)(\vec{k}\cdot\vec{n})A_o^2}{\omega\tilde{D}}\left\{a_o^2(\vec{k}\cdot\vec{b})^2 - \omega^2b^2\right\}.$$

In the limit $\epsilon \to 0$, $\vec{k}\cdot\vec{n} \to 0$ and $b^2 - 1 \to 0$, so that the terms with denominator $\Delta\tilde{D}$ disappear and the solutions

go over to those in I, that is I(4) and I(6). In (10a) and (10b) only the divergence-free part of $\vec{S}_B$, $\vec{k}\times\vec{S}_B$, appears. In (10c) and (10d) we have split the expressions into physical and "unphysical" terms. The unphysical ones have the factor $k\cdot S_B$. In (10c) the remaining physical terms involve only the combinations $\vec{k}\times\vec{S}_B$ or $\vec{S}_B\cdot\vec{j}$ and in (10d) on rewriting $(\vec{S}_B\cdot\vec{j})(\vec{k}\cdot\vec{n}) = (\vec{S}_B\times \vec{k}\cdot\vec{b})+(\vec{k}\cdot\vec{j})(\vec{S}_B\cdot\vec{n})$ only $\vec{k}\times\vec{S}_B$ or $\vec{n}\cdot\vec{S}_B$ appear. Thus we may parallel the arguments of I and assert we have made the separation into physical and unphysical terms.

We may evaluate the terms in the transforms with the denominator Δ directly. We express those with denominator $\tilde{D}$ quite simply in terms of the functions in I. If we set

$$\text{(11a)} \qquad \tilde{a}_o^2 = \frac{a_o^2 + A_1^2}{2} + \frac{1}{2}\sqrt{(a_o^2 + A_1^2)^2 - 4a_o^2\, A_o^2}$$

$$\text{(11b)} \qquad \tilde{A}_o^2 = \frac{a_o^2 + A_1^2}{2} - \frac{1}{2}\sqrt{(a_o^2 + A_1^2)^2 - 4a_o^2\, A_o^2}$$

then $\tilde{D}(\omega,\vec{k}) = D(\omega,\vec{k},\tilde{a}_o,\tilde{A}_o)$, where D refers to the denominator considered in I. It is again possible to show that the discontinuity planes induced by the double points of the normal speed locus do not appear and we may carry over formulas I(11) through I(22) with the replacement (11), and thus functions $\tilde{G}_{xx}$, $\tilde{G}_{xy}$, $\tilde{G}_{yy}$ are defined. The singularity surfaces and lacunas for this case given by I(19), I(20) after the modification (11); thus they are similar to those in I. The

"unphysical" terms are again similar to those in I; as before we shall not include them in the final answers. We must next consider the more complicated transforms with denominator $\Delta\tilde{D}$.

The methods used in I for the explicit evaluation of a class of Fourier transforms required that the functions (or transforms) depend non-trivially on three independent variables. Although the Alfven wave depends on two variables, in I it was independent of the longitudinal wave so we could determine it separately. Now there are terms in the Green's function, i.e., those with denominator $\Delta\tilde{D}$, which are coupled Alfven-longitudinal waves and we need some method of handling them. We shall go through a circuitous procedure to show that the finite part interpretation of a singular integral is again appropriate. We shall proceed to make a series of formal manipulations of the transforms until we can recognize the answer, and finally we shall recompose the answer.

By the simple identity

$$\frac{A_o^2\,\omega^2}{A\tilde{D}} = \frac{(-1)}{(\vec{k}\times\vec{b}\cdot\vec{k}\times b)\Delta} + \frac{\omega^2 - a_o^2\,k^2}{(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{b})\tilde{D}}$$

we are able to make a formal decomposition of the Alfven and longitudinal waves. We study a transform of the form

$$\text{(12a)}\qquad T = \frac{A_o^2\,\omega^{2+\gamma}\,(k_x)^{\alpha}(k_y)^{\beta}}{\Delta\tilde{D}} \qquad \begin{cases} \alpha+\beta+\gamma = 2 \\ \beta \leq 1 \\ \gamma \leq 1 \end{cases}$$

$$\text{(12b)}\qquad T = \frac{-\omega^{\gamma}(k_x)^{\alpha}(k_y)^{\beta}}{(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{b})\Delta} + \frac{\omega^{\gamma}(k_x)^{\alpha}(k_y)^{\beta}(\omega^2 - a_o^2k^2)}{(\vec{k}\times\vec{b}\cdot\vec{k}\times\vec{b})\tilde{D}}$$

(12c) $\quad T = T_A + T_L.$

All the transforms in (10) with denominator $\Delta\tilde{D}$ are first derivatives of the functions whose transforms are considered in (12a). We consider first T_A, the simpler of the two terms.

For $\gamma = 1$ we may write

$$T_A = \frac{(-1)}{(2\pi)^3} \left(-i \frac{\partial}{\partial x}\right)^{\alpha} \left(-i \frac{\partial}{\partial y}\right)^{\beta} \int d\omega \int (dk) \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{2(k^2b^2-(\vec{k}\cdot\vec{b})^2)} \left\{ \frac{1}{\omega - k_x A_o} + \frac{1}{\omega + k_x A_o} \right\}, \qquad \alpha + \beta = 1 .$$

For $\gamma = 0$, so that $\alpha \geq 1$, we may write

$$T_A = \frac{(-1)}{(2\pi)^3} \left(-i \frac{\partial}{\partial x}\right)^{\alpha-1} \left(-i \frac{\partial}{\partial y}\right)^{\beta} \int d\omega \int (dk) \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{2A_o(k^2b^2-(\vec{k}\cdot\vec{b})^2)} \left\{ \frac{1}{\omega - k_x A_o} - \frac{1}{\omega + k_x A_o} \right\}, \qquad \alpha + \beta = 2.$$

Thus it suffices to consider

$$S^{\pm} = \left(\frac{\partial}{\partial x}\right)^{\alpha'} \left(\frac{\partial}{\partial y}\right)^{\beta'} \left(\frac{-i}{(2\pi)^3}\right) \int d\omega \int (dk) \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{((k\cdot b)^2-(\vec{k}\cdot\vec{b})^2)} \left\{ \frac{1}{\omega - k_x A_o} \pm \frac{1}{\omega + k_x A_o} \right\}, \qquad \alpha' + \beta' = 1.$$

We note that

$$\left\{\epsilon^2\left(\frac{\partial}{\partial x}\right)^2+(1+\epsilon^2)\left(\frac{\partial}{\partial y}\right)^2\right\} S^{\pm} = H(t)\left(\frac{\partial}{\partial x}\right)^{\alpha'}\left(\frac{\partial}{\partial y}\right)^{\beta'}\left\{\delta(x-A_o t)\delta(y)\right.$$

$$\left.\pm\ \delta(x+A_o t)\delta(y)\right\} .$$

An elementary calculation gives

$$S^{\pm} = \left(\frac{\partial}{\partial x}\right)^{\alpha'}\left(\frac{\partial}{\partial y'}\right)^{\beta'}\frac{H(t)}{4\pi\epsilon\sqrt{1+\epsilon^2}}\left\{\log\left(\left(\frac{x-A_o t}{\epsilon}\right)^2+\frac{y^2}{1+\epsilon^2}\right)\right.$$

$$\left.\pm\ \log\left(\left(\frac{x+A_o t}{\epsilon}\right)^2+\frac{y^2}{(1+\epsilon^2)}\right)\right\} .$$

$S^{\pm}$ is accordingly a rational homogeneous function of degree -1 in x,y,t. We need not evaluate $S^{\pm}$ explicitly as a general argument will show it does not appear in the answer.

On the other hand T_L satisfies the following equation :

$$(13)\qquad \left(\left(\frac{\partial}{\partial t}\right)^4-\left(\frac{\partial}{\partial t}\right)^2\nabla^2(a_o^2+A_1^2)+\nabla^2\left(\frac{\partial}{\partial x}\right)^2 a_o^2A_o^2\right)\Gamma_L =$$

$$\left(i\frac{\partial}{\partial t}\right)^{\gamma}\left(-i\frac{\partial}{\partial x}\right)^{\alpha}\left(-i\frac{\partial}{\partial y}\right)^{\beta}\left(\left(i\frac{\partial}{\partial t}\right)^2\right.$$

$$\left.-a_o^2(-i\nabla)^2\right)\frac{(-1)\delta(t)}{4\pi\epsilon\sqrt{1+\epsilon^2}}\log\left(\left(\frac{x}{\epsilon}\right)^2+\frac{y^2}{1+\epsilon^2}\right) \qquad \begin{array}{l}\alpha+\beta+\gamma=2\\ \alpha+\gamma\geq 1\end{array}$$

Just as in Section III of I, if we determine T_L from its

Fourier transform and then do the $|k|$ and ω integrals we obtain

$$(14)\quad T_L = \frac{H(t)}{2\pi^2 t}\int_0^{\pi} d\theta\, \frac{(\hat{k}_x)^{\alpha}(\hat{k}_y)^{\beta}(\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)}{\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t}, \hat{k})((\hat{k}_x)^2\epsilon^2+(1+\epsilon^2)(\hat{k}_y)^2)} .$$

Again we come to the problem of the interpretation of singular integrals. The finite part interpretation of the integral is appealing, but if we were to go through the plane wave techniques suggested in I, we would learn that again special irregularities associated with the multiple roots of $\tilde{D}$ might appear. But for this system (characterized by $\alpha+\gamma \geq 1$) the irregularities do not occur and we may write, with the notation of I:

$$(15)\quad T_L = \frac{H(t)}{4\pi^2 t}\int_{C+C'} d\theta\, \frac{(\hat{k}_x)^{\alpha}(\hat{k}_y)^{\beta}(\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)}{\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t}, \hat{k})((\hat{k}_x)^2\epsilon^2+(1+\epsilon^2)(\hat{k}_y)^2)} .$$

At the points $\epsilon^2(\hat{k}_x)^2 + (1+\epsilon^2)(\hat{k}_y)^2 = 0$, or at $\cos^2\theta = (1+\epsilon^2)$, the second factor in the denominator vanishes. Let $\tilde{C}$ be a curve oriented in a clockwise sense that surrounds the values $\cos^2\theta = 1+\alpha^2$ in the upper half plane in the strip $0 \leq \mathrm{Re}\ \theta \leq \pi$ and let $\tilde{C}$ surround no root of $\tilde{D}$, unless it coincides with the selected values. Let $\tilde{C}'$ be a similar contour in the lower half plane but oriented in a counter clockwise sense. Thus we have

$$(16)\quad T_L = \frac{H(t)}{4\pi^2 t}\int_{(C+\tilde{C})+(C'+\tilde{C}')} d\theta \; \frac{(\hat{k}_x)^{\alpha}(\hat{k}_y)^{\beta}(\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)}{\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t},\ \hat{k})(\epsilon^2(\hat{k}_x)^2+(1+\epsilon^2)(\hat{k}_y)^2)}$$

$$- \frac{H(t)}{4\pi^2 t}\int_{\tilde{C}+\tilde{C}'} d\theta \; \frac{(\hat{k}_x)^{\alpha}(\hat{k}_y)^{\beta}(\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)}{\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t},\ \hat{k})(\epsilon^2(\hat{k}_x)^2+(1+\epsilon^2)(\hat{k}_y)^2)} \; .$$

On performing the second integration we obtain a rational homogeneous function of degree -1 in x,y,t. The first term, as we shall see below, vanishes outside some bounded region of the x/t, y/t plane. We expect $T = T_A + T_L = T_L + C^{\pm} S^{\pm}$, with $C^{\pm}$ some constants, the fundamental solution of a hyperbolic partial differential equation, to vanish identically outside some bounded set. Thus T_A and the second integral in (16) must cancel in this region, but then they must cancel everywhere so that we may write finally:

$$(17)\quad T = \frac{H(t)}{4\pi^2 t}\int_{(C+\tilde{C})+(C'+\tilde{C}')} \frac{(\hat{k}_x)^{\alpha}(\hat{k}_y)^{\beta}(\frac{\hat{k}\cdot\vec{r}}{t})^{\gamma}((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)}{\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t},\ \hat{k})(\epsilon^2(\hat{k}_x)^2+(1+\epsilon^2)(\hat{k}_y)^2)} \; .$$

We might readily verify that (17) is just the result obtained from a finite part interpretation of the formal Fourier inversion of (12a) by the methods of Section III of I, but we shall leave the answer in its present form.

In I the change of variable $z = e^{2i(\theta-\phi)}$ transformed the integral I(12) into an integral of a rational function around

a closed contour, I(13),(14), expressible in terms of the residues at the poles enclosed by the contour (the poles occuring only at the complex roots of $\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t}, \hat{k}))$ I(22). Here too the transformation $z = e^{2i(\theta-\emptyset)}$ turns (17) into the integral around a closed contour of a rational function whose only poles inside the contour occur at the complex roots of $\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t}, \hat{k})$. We might write explicit forms for the integral analogous to I(22), but as they are not particularly illuminating we shall omit them. We define the value of (17) to be $H_{xx}(x,y,t)$, $\alpha = 2, \beta = \gamma = 0$; $H_{xy}(x,y,t)$, $\alpha = \beta = 1, \gamma = 0$; $H_{yy}(x,y,t)$, $\beta = 2, \alpha = \gamma = 0$.

For (17) we may carry over essentially all the general discussion of Section IV of I always with the modification implied by (11) included. Thus we have as at least part of the ray cone, the surface given by I(19), (20) (with the substitution (11)). Again (see Figure 1) a section of the ray cone consists of an outer region A which the disturbance has not yet reached, the region B of the disturbance, and the lacunas C. There is, however, an addition to the ray cone coming from the singularity produced as the roots of $\tilde{D}(\frac{\hat{k}\cdot\vec{r}}{t}, \hat{k})$ approach those of $\epsilon^2 k_x^2 + (1+\epsilon^2)k_y^2$. A direct calculation (depending on the fact that $((\frac{\hat{k}\cdot\vec{r}}{t})^2 - a_o^2)$ occurs in the numerator in (17)) shows that the only relevant confluence of roots occurs at $x^2 = A_o^2 t^2$, $y = 0$, the position of a pure Alfven wave. Thus there are added singularity points in Figure 1, the points E,E' where a pure Alfven wave would

be. The integral (17) has the character of a compound of an Alfven and a longitudinal wave, just as it should. A more complete analysis would also show that the singularity in a derivative of T near E or E' is the same as a pure Alfven wave, i.e., it is of the form $\delta(x \pm At)\delta(y)$.

We now give the explicit forms for the physical parts of the Green's function:

$$(18a)\quad \rho = (-\frac{\partial}{\partial t})S_\rho\left\{((\frac{x}{t})^2 - A_1^2)\tilde{G}_{xx} + \frac{2xy}{t^2}\tilde{G}_{xy} + \frac{y^2}{t^2}\tilde{G}_{yy}\right\}$$

$$+ \frac{\partial}{\partial x} S_{u_x}\left\{((\frac{x}{t})^2 - A_o^2)\tilde{G}_{xx} + \frac{2xy}{t^2}\tilde{G}_{xy} + \frac{y^2}{t^2}\tilde{G}_{yy}\right\}$$

$$-(\frac{\partial}{\partial y} A_1^2 S_\rho + \frac{\partial}{\partial t} S_{u_y})(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}) - \frac{\partial}{\partial y} A_o^2(\vec{b}\cdot\vec{S}_u)\tilde{G}_{xy}$$

$$- \frac{\partial}{\partial x}\epsilon A_o^2 S_{u_z}\tilde{G}_{xx} + (\vec{\nabla}\times\vec{S}_B)\cdot\left\{\vec{j} A_o^2\epsilon(\frac{x}{t}\tilde{G}_{xx} + \frac{y}{t}\tilde{G}_{xy})\right.$$

$$\left. + \vec{n}A_o^2(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy})\right\}$$

$$(18b)\quad (\vec{u}\cdot\vec{b}) = (\frac{\partial}{\partial x} S_\rho - \frac{\partial}{\partial t}(\vec{b}\cdot\vec{S}_u))((\frac{x^2}{t^2} - A_1^2)\tilde{G}_{xx} + \frac{2xy}{t^2}\tilde{G}_{xy} + \frac{y^2}{t^2}\tilde{G}_{yy})$$

$$+ a_o^2(\vec{\nabla}\times\vec{S}_u)\cdot\left\{(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy})\vec{n} + \epsilon\vec{j}(\frac{x}{t}\tilde{G}_{xx} + \frac{y}{t}\tilde{G}_{xy})\right\}$$

$$- \frac{\partial}{\partial y} A_1^2\, \vec{b}\cdot\vec{S}_u(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}) - \frac{\partial}{\partial y} A_1^2 S_\rho\tilde{G}_{xy}$$

$$+ a_o^2 A_o^2(\vec{\nabla}\times\vec{S}_B)\cdot(-\epsilon\vec{j}\,\tilde{G}_{xx} - \vec{n}\,\tilde{G}_{xy})$$

$$(18c)\quad (\vec{B}\cdot\vec{b}) = -\left(\frac{\partial}{\partial x}(\vec{S}_u\cdot\vec{b})+(\vec{S}_B\cdot\vec{b})\left(+\frac{\partial}{\partial t}\right)\right)\left\{\left(\left(\frac{x}{t}\right)^2 - a_o^2\right)\tilde{G}_{xx}\right.$$

$$\left. + \frac{2xy}{t^2}\tilde{G}_{xy} + \left(\frac{y}{t}\right)^2\tilde{G}_{yy}\right\} + (\vec{S}_u\cdot\vec{\nabla})\left\{\left(b^2\left(\frac{x}{t}\right)^2 - a_o^2\right)\tilde{G}_{xx}\right.$$

$$\left. + \frac{2b^2xy}{t^2}\tilde{G}_{xy} + \frac{2b^2y^2}{t^2}\tilde{G}_{yy}\right\} - \frac{\partial}{\partial y} a_o^2(\vec{S}_B\cdot\vec{b})\left(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}\right)$$

$$+ S_\rho a_o^2\left\{\epsilon^2\frac{\partial}{\partial x}\left(\frac{x}{t}\tilde{G}_{xx} + \frac{y}{t}\tilde{G}_{yy}\right) + (1+\epsilon^2)\frac{\partial}{\partial y}\left(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}\right)\right\}$$

$$+ a_o^2(\vec{b}\cdot\vec{S}_u)\frac{\partial}{\partial y}\tilde{G}_{xy}.$$

$$(18d)\quad (\vec{u}\cdot\vec{j}) = \left(A_o^2(\vec{n}\cdot\vec{\nabla}\times\vec{S}_B) - \frac{\partial}{\partial t}(\vec{S}_u\cdot\vec{j})\right)\left(\left(\left(\frac{x}{t}\right)^2 - a_o^2\right)\tilde{G}_{xx}\right.$$

$$\left. + \frac{2xy}{t^2}\tilde{G}_{xy} + \frac{y^2}{t^2}\tilde{G}_{yy}\right) + a_o^2\left(S_{u_x}\frac{\partial}{\partial x} - \frac{\partial}{\partial t}S_\rho\right)\left(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}\right)$$

$$- a_o^2A_o^2\epsilon(\vec{b}\cdot\vec{\nabla}\times\vec{S}_B)\tilde{G}_{xx} - \epsilon(\vec{b}\cdot\vec{\nabla}\times\vec{S}_u)\left(\frac{x}{t}H_{xx} + \frac{y}{t}H_{xy}\right)$$

$$+ \epsilon A_o^2(\vec{b}\cdot\vec{\nabla}\times\vec{S}_B)H_{xx}$$

$$(18e)\quad (\vec{u}\cdot\vec{n}) = H(t)\frac{\delta(y)}{2}\left(((\vec{S}_u\cdot\vec{n}) - A_oS_{B_z})\delta(x-A_ot)\right.$$

$$\left. +(\vec{S}_u\cdot\vec{n}) + A_oS_{B_z})\delta(x+A_ot)\right) + \epsilon^2(\vec{j}\cdot\vec{\nabla}\times\vec{S}_B)\left(\left(\frac{x}{t}\right)^2\tilde{G}_{xx} + \frac{2xy}{t^2}\tilde{G}_{xy}\right.$$

$$\left. +\left(\frac{y}{t}\right)^2\tilde{G}_{yy}\right) + \epsilon a_o^2(\vec{n}\cdot\vec{\nabla}\times\vec{S}_B)\tilde{G}_{xy} - \epsilon\, a_o^2\left(\vec{\nabla}\cdot\vec{S}_u - \frac{\partial}{\partial t}S_\rho\right)\left(\frac{x}{t}\tilde{G}_{xx}\right.$$

$$\left. + \frac{y}{t}\tilde{G}_{xy}\right) + \epsilon^2(\vec{j}\cdot\vec{\nabla}\times\vec{S}_B)H_{xx} + \epsilon b^2(\vec{n}\cdot\vec{\nabla}\times\vec{S}_B)H_{xy}$$

$$- \epsilon(\vec{b}\times\vec{\nabla}\cdot\vec{b}\times\vec{S}_u)\left(\frac{x}{t}H_{xx} + \frac{y}{t}H_{xy}\right)$$

$$(18f)\quad (\vec{B}\cdot\vec{j}) = -\left((\vec{S}_B\cdot\vec{j})\frac{\partial}{\partial t} + (\vec{S}_u\cdot\vec{j})\frac{\partial}{\partial x}\right)\left(\left(\left(\frac{x}{t}\right)^2 - a_o^2\right)\tilde{G}_{xx} + \frac{2xy}{t^2}\tilde{G}_{xy}\right.$$

$$\left. + \left(\frac{y}{t}\right)^2\tilde{G}_{yy}\right) + a_o^2\left(\frac{\partial}{\partial t} S_\rho - S_{u_x}\frac{\partial}{\partial x}\right)\tilde{G}_{xy}$$

$$- a_o^2\frac{\partial}{\partial y}(\vec{S}_B\cdot\vec{j})\left(\frac{x}{t}\tilde{G}_{xy} + \frac{y}{t}\tilde{G}_{yy}\right) + (\vec{b}\cdot\vec{\nabla}\times\vec{S}_B)\left(\frac{x}{t}H_{xx} + \frac{y}{t}H_{xy}\right)\epsilon$$

$$+ \epsilon(\vec{b}\cdot\vec{\nabla}\times\vec{S}_u)H_{xx}$$

$$(18g)\quad (\vec{B}\cdot\vec{n}) = H(t)\,\frac{\delta(y)}{2A_o}\left\{(A_oS_{B_z} - \vec{S}_u\cdot\vec{n})\delta(x-A_ot)\right.$$

$$\left. + (A_oS_{B_z} + \vec{S}_u\cdot\vec{n})\delta(x+A_ot)\right\} + \epsilon a_o^2\left(\vec{\nabla}\cdot\vec{S}_u - \frac{\partial}{\partial t}S_\rho\right)\tilde{G}_{xx}$$

$$+ \epsilon(\vec{b}\times\vec{\nabla}\cdot\vec{b}\times\vec{S}_u)H_{xx} + \left(\epsilon^2(\vec{S}_B\cdot\vec{n})\frac{\partial}{\partial x} - \epsilon\frac{\partial}{\partial y}(\vec{S}_B\cdot\vec{j})\right)\left(\frac{x}{t}H_{xx}\right.$$

$$\left. + \frac{y}{t}H_{xy}\right) + \epsilon^2\frac{\partial}{\partial y}(\vec{S}_B\cdot\vec{n})\left(\frac{x}{t}H_{xy} + \frac{y}{t}H_{yy}\right).$$

Again all derivatives operate on all functions to the right.

With reference to Figure 1, all the functions $\tilde{G}$, H vanish in regions A and C and become infinite as one approaches the boundary of B. On the boundary of B the singularity is of highest order at D and D'. For the functions $\tilde{G}$ there are no other singularities. The functions H are also singular at E, E', the position of the Alfven wave, and behave there like a pure Alfven wave.

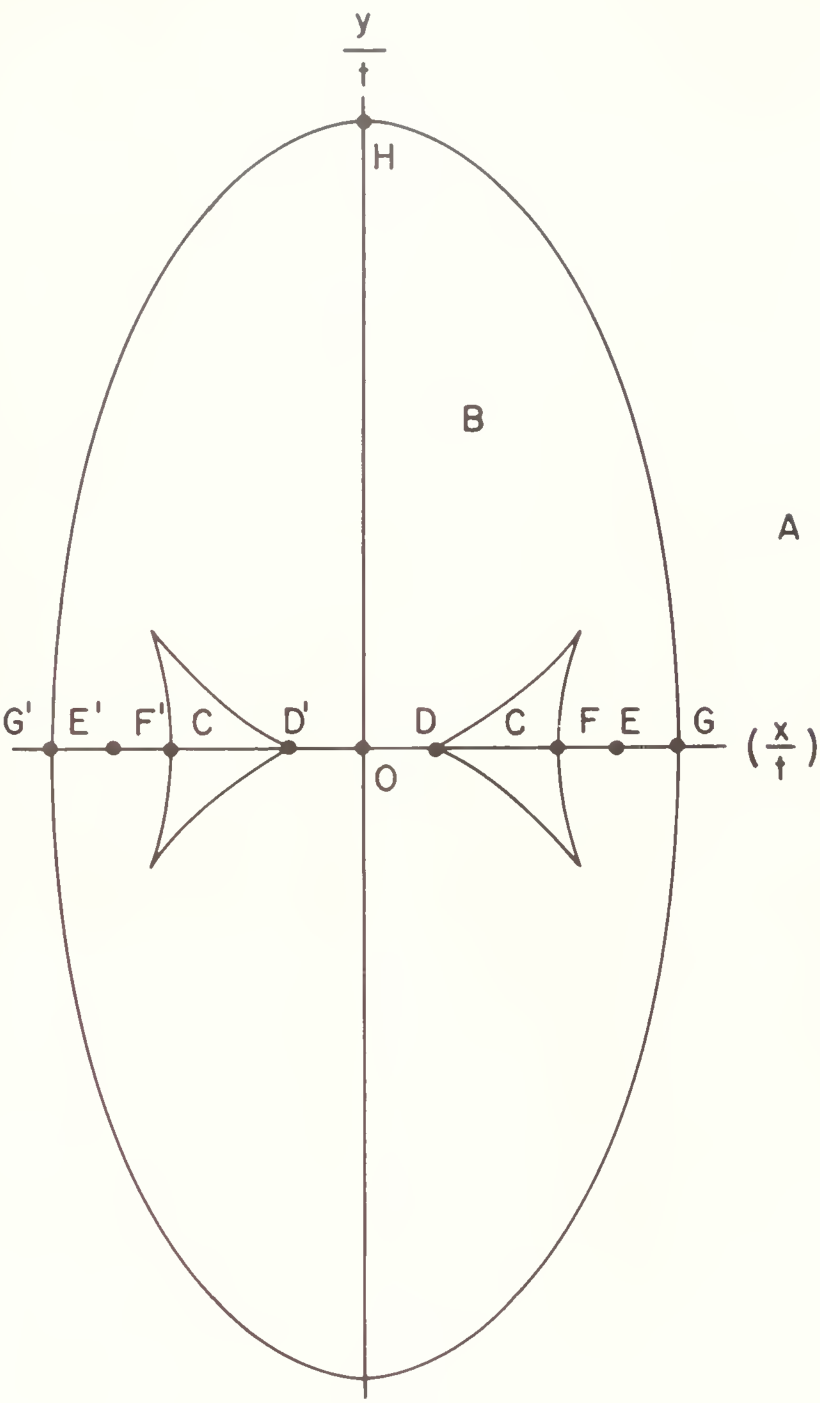
FOOTNOTES

1. H. Weitzner, "On the Green's Function for Two Dimensional Magnetohydrodynamic Waves I", NYO-2886, New York University April, 1960. We shall refer to this report as I and use the same notation here.

2. A preliminary account of these results was presented at the November 1960 meeting of the Division of Fluid Dynamics of the American Physical Society at Baltimore, Maryland.

3. See A. A. Blank and H. Grad, "Notes on Magnetohydrodynamics VII - Fluid Dynamic Analogies", NYO-6486, New York University July, 1958.

4. H. Grad in <u>The Magnetodynamics of Conducting Fluids</u>, ed., D. Bershader, Stanford University Press, 1959.



$$OD = OD' = \sqrt{\frac{a_0^2 A_0^2}{a_0^2 + A_1^2}}$$

$$OF = OF' = \tilde{A}_0$$

$$OE = OE' = A_0$$

$$OG = OG' = \tilde{a}_0$$

$$OH = \sqrt{a_0^2 + A_1^2}$$

FIG. 1

SECTION OF THE RAY CONE FOR A TWO DIMENSIONAL MAGNETOHYDRODYNAMIC WAVE

NYU c.1
NYO-9489
Weitzner

AUTHOR
On the Green's function for

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